Lecture 6 **2012/11/05**

Outline

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1. Motivation

- A particular class of system with global existence. (**Fundamental Theorem**)
- A good approximation to nonlinear system near a known solution. (**Linearization**)
- Solution structure and geometric property of the set of whole solutions. (**Solution Structure and Direct Sum Decomposition**)
- The theory of linear system is relatively complete except a few remains open.

If $x = x_0(t)$ is a particular solution of the following nonlinear systems

$$x' = f(t, x), \tag{NS}$$

where f is continuously differentiable, then $y = x - x_0(t)$ in (NS) implies

$$y' = x' - x_0'(t) = f(t, x) - f(t, x_0(t)) = f(t, y + x_0(t)) - f(t, x_0(t))$$

$$= A(t)y + \alpha(t, y), \tag{LN1}$$

where $A(t) = \dfrac{\partial f}{\partial x}(t, x_0(t))$, $\dfrac{\partial f}{\partial x}(t, y) \neq 0$ near $y = 0$, $\alpha(t, 0) = 0$ and $\dfrac{\partial \alpha(t, 0)}{\partial x} = 0$.

It is natural to study the linear system

$$z' = A(t)z, \tag{LN2}$$

and relationship between $y(t)$ of (LN1) and $z(t)$ of (LN2) near $y = 0$. That is, is (LN2) a good approximation of (LN1) near $y = 0$? What dose the condition $\dfrac{\partial f}{\partial x}(t, y) \neq 0$ near $y = 0$ imply? (**Hartman-Grobman Theorem)**

2. Global Existence

Consider

$$\dot{x} = A(t)x + h(t)\,, \quad x \in R^n\,,$$

where $A(t)$ and $h(t) \in C(I)$, $I = (a, b)$ ($a = -\infty$ or $b = \infty$ is permitted). $\Rightarrow$ The following IVP

$$\dot{x} = A(t)x + h(t)\,, \quad x(t_0) = x_0 \tag{L1}$$

has a local existence and uniqueness by Picard theorem. But the IVP (L1) has a global result!

Theorem 6.1 ((Fundamental Theorem) Suppose that $A(t)$ and $h(t) \in C(I)$, where $I = (a,b)$, if $t_0 \in I$, then the solution of the IVP (L.1) is defined on (a,b).

Proof. Since $f(t,x) = A(t)x + h(t) \in C(I \times R^n)$ satisfies locally Lipschitz, then there exists a unique solution $x(t, t_0, x_0)$ of the IVP (L1) by Picard theorem, defined on $I_{\max}$. If $I_{\max} \neq (a,b)$, say $I_{\max}^{+} = [t_0, \omega_+)$ with $\omega_+ < b$, we show that by contradiction.

Let $t \in [t_0, \omega_+)$. Then we have

$$x(t,t_0,x_0) - x_0 = \int_{t_0}^{t} \{A(s)x(s, t_0, x_0) + h(s)\}\,ds\,.$$

$$\Rightarrow \quad \| x(t,t_0,x_0) \| \le \| x_0 \| + \int_{t_0}^{t} \| A(s) \| \| x(s, t_0, x_0) \| \,ds + \int_{t_0}^{\omega_+} \| h(s) \| \,ds$$

By Gronwall's inequality, we have

$$\begin{aligned} \| x(t,t_0, x_0) \| &\le \{\| x_0 \| + \int_{t_0}^{\omega_+} \| h(s) \| \,ds\} \cdot \exp(\int_{t_0}^{t} \| A(s) \| \,ds) \\ &\le \{\| x_0 \| + \int_{t_0}^{\omega_+} \| h(s) \| \,ds\} \cdot \exp(\int_{t_0}^{\omega_+} \| A(s) \| \,ds) < \infty\,. \end{aligned}$$

Thus, $\| x(t,t_0, x_0) \|$ is uniformly bounded for (t_0, x_0) on $I_{\max}^{+}$. However, we obtain $\| x(t,t_0, x_0) \| \to \infty$ as $t \to \omega_+^{-}$ by the continuation theorem. This is a contradiction. It is similar to show the case of $I_{\max}^{-} = (\omega_-, t_0]$ with $\omega_- > a$. □

Remark 6.1 Linear system has global solutions. We can regard whole solutions of the linear system (F1) as a set just because of global existence.

3. Superposition Principle

Theorem 6.2 (Superposition Principle) Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h_1(t)$ and $x' = A(t)x + h_2(t)$ respectively. Then $c_1x_1(t) + c_2x_2(t)$ is a solution of $x' = A(t)x + (c_1h_1(t) + c_2h_2(t))$.

Remark 6.2 From math point of view, superposition principle is quite simple. However, it is a characterization of linear systems. Extremely important in practice! From control point of view, if we regard $h(t)$ as an input and $x(t)$ response (output), then, any dynamic system satisfies superposition principle is linear. This principle can be verified based on experiment without having differential equations.

Corollary 6.1 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x$. Then $c_1x_1(t) + c_2x_2(t)$ is a solution of $x' = A(t)x$.

Proof. Taking $h_1(t) = h_2(t) \equiv 0$ in Theorem 6.2 yields the result. □

Corollary 6.2 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h(t)$ and $x' = A(t)x$ respectively. Then $x_1(t) + x_2(t)$ is a solution of $x' = A(t)x + h(t)$.

Proof. Taking $h_1(t) = h(t)$, $h_2(t) \equiv 0$ in Theorem 6.2 yields the result. □

Corollary 6.3 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h(t)$. Then $x_1(t) - x_2(t)$ is a solution of $x' = A(t)x$.

Proof. Taking $h_1(t) = h_2(t) = h(t)$ in Theorem 6.2 yields the result. □

4. Homogeneous Linear Systems

Consider the homogeneous linear system $x' = A(t)x$. Let Ω be the set of whole solutions of $x' = A(t)x$.

Theorem 6.3 Ω is an n-dimensional vector linear space.

Proof. For any $t_0 \in I$ and $x_0 \in R^n$, there exists a unique solution $x(t, x_0)$, $t \in I$. Define a mapping $T: R^n \to \Omega$ as follows.

$$T(x_0) = x(t, x_0).$$

We first show that T is a linear mapping. We know by the superposition principle that $c_1 x(t, x_0^1) + c_2 x(t, x_0^2) \in \Omega$ if $x(t, x_0^1), x(t, x_0^2) \in \Omega$ and it satisfies the initial value condition $c_1 x_0^1 + c_2 x_0^2$. By uniqueness,

$$x(t, c_1 x_0^1 + c_2 x_0^2) = c_1 x(t, x_0^1) + c_2 x(t, x_0^2).$$

Then,

$$T(c_1 x_0^1 + c_2 x_0^2) = x(t, c_1 x_0^1 + c_2 x_0^2) = c_1 x(t, x_0^1) + c_2 x(t, x_0^2) = c_1 T(x_0^1) + c_2 T(x_0^2).$$

Therefore T is a linear mapping.

We then show that T is an isomorphic mapping. For any $x(t) \in \Omega$, there exists $x_0 \in R^n$ such that $T(x_0) = x(t)$. Therefore, T is onto. Next for any $x(t, x_0^1)$, $x(t, x_0^2) \in \Omega$ with $x(t, x_0^1) \neq x(t, x_0^2)$, it must be $x_0^1 \neq x_0^2$. It is also true inversely. Therefore, T is one-to-one. Combining these two, T is isomorphic $\Rightarrow$ $\Omega \cong R^n$. □

Remark 6.3 Geometric meaning of $T: R^n \to \Omega$: any integral curve $x(t) \in \Omega$ always intersects uniquely the super-plane $t = t_0$ at some point $x_0 \in R^n$.

Remark 6.4 Since $\Omega \cong R^n$, the algebraic structure of Ω is clear. Any n linearly independent elements (solutions, vector functions) of Ω form a base of Ω.

Definition 6.1 $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ ($t \in I$) is said to be **linearly independent** in I if $\sum_{j=1}^{n} c_j x_j(t) \equiv 0$ for all $t \in I$ implies that $c_1 = c_2 = \cdots c_n = 0$. Conversely, these vector-valued functions are said to be **linearly dependent** in I if there exist

$c_1, c_2, \cdots, c_n$ not all zero s.t. $\sum_{j=1}^{n} c_j x_j(t) \equiv 0$ for all $t \in I$.

Definition 6.2 Any solutions $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ ($t \in I$) that are linearly independent is said to be **a fundamental set of solutions**.

Example 6.1 Show that $n+1$ vector-valued functions

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R^n, x_2(t) = \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R^n, \cdots, \text{ and } x_{n+1}(t) = \begin{pmatrix} t^n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R^n$$

are linearly independent on any interval I.

Show by contradiction. If there exist $c_1, c_2, \cdots, c_{n+1}$ not all zero s.t. $\sum_{j=1}^{n+1} c_j x_j(t) \equiv 0$ for all $t \in I$, then we have $c_1 + c_2 t + \cdots + c_{n+1} t^n \equiv 0$ for all $t \in I$. But there are at most n roots only for this polynomial equation according to the fundamental theorem of algebra unless $c_1 = c_2 = \cdots = c_{n+1} = 0$. This is a contradiction.

Remark 6.5 This example seems contradiction to Theorem 6.3 which says that there only exist n linearly independent vector-valued solutions. In fact, it is not. Why?

Theorem 6.4 (General Solution Structure) Let $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ be a fundamental set of solutions. Then the general solution is

$$x(t) = \sum_{j=1}^{n} c_j x_j(t),$$

where $c_1, c_2, \cdots, c_n$ are any arbitrary constants. Moreover, it includes the whole solutions.

Proof. Take a base of R^n as follows.

$$x_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad x_n(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The corresponding solutions are as follows.

$$x_1(t), x_2(t), \cdots, x_n(t) \in \Omega, \quad t \in I.$$

It remains to show that $x_1(t), x_2(t), \cdots, x_n(t)$ is a fundamental set of solutions. If

$$\sum_{j=1}^{n} c_j x_j(t) \equiv 0, \quad t \in I,$$

there exists a mapping $T : R^n \to \Omega$ such that $T\{x_j(t_0)\} = x_j(t)$ ($j = 1, 2, \cdots, n$) by Theorem 6.3. Then

$$\sum_{j=1}^{n} c_j T\{x_j(t_0)\} = 0.$$

Since T is linear, we have $T\{\sum_{j=1}^{n} c_j x_j(t_0)\} = 0$. Then we have $\sum_{j=1}^{n} c_j x_j(t_0) = 0$. $\Rightarrow$ $c_1 = c_2 = \cdots = c_n = 0$. Therefore, $x_1(t), x_2(t), \cdots, x_n(t)$ are linearly independent in I.

We conclude that $x(t) = \sum_{j=1}^{n} c_j x_j(t)$ is a general solution. Moreover, it includes the whole solution because for $\forall\, x(t) \in \Omega$ with $x(t_0)$, there exist c_j^0, $j = 1, 2, \cdots, n$ s.t. $x(t_0) = \sum_{j=1}^{n} c_j^0 x_j(t_0)$ since $\{x_j(t_0)\}$ is a basis of R^n. Then, $x(t) \equiv \sum_{j=1}^{n} c_j^0 x_j(t)$ by uniqueness. This shows that the general solution includes the whole solution. □

Definition 6.3 Let $\Phi(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. $\Phi(t)$ is said to be **a matrix solution** if $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$**.** $\Phi(t)$ is said to be **a fundamental matrix solution** if $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ are linearly independent. A matrix solution $\Phi(t)$ is said to be **a principle matrix solution** if $\Phi(t_0) = I_n$, which is denoted as $\Phi(t, t_0)$.

Theorem 6.5 (Matrix Form of General Solution) The general solution of $x' = A(t)x$ is $x(t) = \Phi(t)c$, where $\Phi(t)$ is a fundamental matrix solution and $c = (c_1, c_2, \cdots, c_n)^T$ is an arbitrary n vector. The solution of the IVP $x' = A(t)x$ with $x(t_0) = x_0$ is

$x(t) = \Phi(t, t_0)x_0$.

Lemma 6.1 A matrix solution $\Phi(t)$ is a fundamental matrix solution $\Leftrightarrow$ $\det\Phi(t) \neq 0$ for all $t \in I$.

Proof. It is noted that $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ are linearly independent

$\Leftrightarrow$ $\det(x_1(t), x_2(t), \cdots, x_n(t)) \neq 0$ for all $t \in I$. □

Definition 6.4 Denote $W(t) = \det\Phi(t)$ is said to be **a Wroskian determinant**.

Remark 6.6 $\Phi(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $t \in (-\infty, \infty)$; but $\det\Phi(t) \equiv 0$, $t \in (-\infty, \infty)$. What is implied from this example?

Theorem 6.6 (Liouville Formula) Suppose that $\Phi(t)$ is a matrix solution of $x' = A(t)x$. Then

$$\det\Phi(t) = \det\Phi(t_0)\exp\{\int_{t_0}^{t} trA(s)ds\},$$

where $trA(t) = \sum_{j=1}^{n} a_{jj}(t)$ is a trace of $A(t)$, $t \in I$.

Proof. It suffices to show that $\det\Phi(t)$ satisfies $\dot{x} = trA(t)x$. Tailor expansion of $\Phi(t+h)$ near $h = 0$ yields

$$\Phi(t+h) = \Phi(t) + h\dot{\Phi}(t) + O(h^2) = (I + hA(t))\Phi(t) + O(h^2).$$

Since $\det(I + hA(t)) = \prod_{i=1}^{n}(1 + ha_{ii}(t)) + O(h^2) = 1 + h \cdot trA(t) + O(h^2)$, we have

$$\frac{\det\Phi(t+h) - \det\Phi(t)}{h} = trA(t)\det\Phi(t) + O(h).$$

Let $h \to 0$ and we obtain that $\det\Phi(t)$ is a solution of $\dot{x} = trA(t)x$. □

Remark 6.7 Liouville Formula implies that if $\Phi(t)$ is a matrix solution of $x' = A(t)x$, then

$$\det\Phi(t) \neq 0 \text{ for all } t \in I \iff \det\Phi(t_0) \neq 0 \text{ for some } t_0 \in I;$$

$$\det\Phi(t) \equiv 0 \text{ for all } t \in I \iff \det\Phi(t_0) = 0 \text{ for some } t_0 \in I.$$

Theorem 6.7 A matrix solution $\Phi(t)$ is a fundamental matrix solution $\iff$ there exists a point $t_0 \in I$ s.t. $\det\Phi(t_0) \neq 0$.

Theorem 6.8 (Properties of $\Phi(t, t_0)$) (Homework)

1) $\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$;

2) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$;

3) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$;

4) $x(t, t_0, x_0) = \Phi(t, t_0)x_0$

4. Non-homogeneous Linear Systems

Consider non-homogeneous linear system

$$\dot{x} = A(t)x + h(t), \quad x(t_0) = x_0.$$

Theorem 6.9 (General Solution Structure for Non-homogeneous Linear Systems) Suppose $x^*(t)$ is a particular solution of $\dot{x} = A(t)x + h(t)$; $\Phi(t)$ is a fundamental matrix solution of its corresponding homogeneous linear system $\dot{x} = A(t)x$. Then the general solution of $\dot{x} = A(t)x + h(t)$ is given by

$$x(t) = \Phi(t)c + x^*(t),$$

where c is an arbitrary vector constant. Moreover it includes the whole solutions.

Proof. By the superposition principle (Corollary 6.2), $\Phi(t)c + x^*(t)$ is a solution of $\dot{x} = A(t)x + h(t)$. Since $\dfrac{\partial x(t)}{\partial c} = \Phi(t)$ is nonsingular for all $t \in I$, $\Phi(t)c + x^*(t)$ is a general solution too. Next, we show that $\Phi(t)c + x^*(t)$ includes the whole solution of $\dot{x} = A(t)x + h(t)$.

For any solution $\tilde{x}(t)$ of $\dot{x} = A(t)x + h(t)$, Take $c_0 = \Phi^{-1}(t_0)(\tilde{x}(t_0) - x^*(t_0))$, it follows that $\tilde{x}(t)$ and $x(t) = \Phi(t)\Phi^{-1}(t_0)(\tilde{x}(t_0) - x^*(t_0)) + x^*(t)$ have the same initial value condition $x(t_0) = \tilde{x}(t_0)$ and so we find c_0 s.t. $x(t) = \Phi(t)c_0 + x^*(t) \equiv \tilde{x}(t)$. This shows that $\tilde{x}(t)$ is an element of the general solutions. □

Remark 6.8 $x^*(t)$ can be determined by $\Phi(t)$ by the method of **Variation of Constants.**

Theorem 6.10 (Variation of Constants) The general solution of $\dot{x} = A(t)x + h(t)$ is given by

$$x(t) = \Phi(t)c + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s\ ;$$

The IVP of $\dot{x} = A(t)x + h(t)$ with $x(t_0) = x_0$ is given by

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s \\ &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)h(s)\,\mathrm{d}s, \end{aligned}$$

where $\Phi(t)$ is a fundamental matrix solution and $\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$ is a principle matrix solution.

Proof. Suppose that $x(t) = \Phi(t)c(t)$ is a solution of $\dot{x} = A(t)x + h(t)$, where $c(t)$ will be determined. Substituting $x(t) = \Phi(t)c(t)$ into $\dot{x} = A(t)x + h(t)$, we have

$$\Phi(t)c'(t) = h(t).$$

Then,

$$c(t) = \int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s\ .$$

From this, it follows that

$$x^*(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s\,.$$

By the general solution structure, it yields

$$x(t) = \Phi(t)c + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s,$$

where c is an arbitrary vector constant. If $x(t_0) = x_0$ is satisfied, $c = \Phi^{-1}(t_0)x_0$ is determined. Then

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s$$

$$= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)h(s)\,\mathrm{d}s. \quad \square$$

6. Summary

- **Linear system has global solutions;**
- **Linear system has superposition principle, which is a key characterization of linear system no matter of equations;**
- **Linear system has an important algebra property: $\Omega \cong R^n$, finite dimension;**
- **Linear system has general solution structure formulae.**
- **How to find a fundamental matrix solution $\Phi(t)$ remains unsolved.**

7. Homework

1. Show that $\dot{x} = A(t)x + h(t)$ has only $n+1$ linearly independent solutions, where $h(t)$ is not identically zero on I; $A(t)$ and $h(t)$ are continuous on I.
2. Show that the IVP

$$\dot{x} = A(t)x + f(t, x), \quad x(t_0) = x_0$$

and the integral equations

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) f(s, x(s))\,\mathrm{d}s$$

are equivalent. That is, they have the same set of solutions, where $\Phi(t)$ is a fundamental matrix solution, $A(t)$ is continuous on I and $f(t, x)$ is continuous on $I \times R^n$.